Abstract: In 2009, the National Council of Teachers of Mathematics (NCTM) in the U.S.A. released a document entitled *Focus in High School Mathematics: Reasoning and Sense Making* as the next step in its efforts to promote high standards in mathematics education. Building on *Principles and Standards for School Mathematics* (NCTM, 2000), this publication argues that a focus on reasoning and sense making will best prepare students for future success. This paper will include a brief overview of the publication and its rationale. It will also provide examples of how reasoning and sense making can become the focus of mathematics instruction across the curriculum.

Founded in 1920, the National Council of Teachers of Mathematics (NCTM) has been a leading voice for improving mathematics education in the United States. Over the past three decades a central emphasis of the organization has been on the importance of “process” skills in mathematics education, along with high-quality content. This emphasis began in 1980 with *An Agenda for Action*, which advocated for making problem solving the center of mathematics education. This emphasis on process was reinforced and extended by the release of the *Curriculum and Evaluation Standards for School Mathematics* in 1989, which proposed more detailed standards for three gradebands (K-4, 6-8, and 9-12), including more detailed recommendations for both the content and process skills that students should develop. In addition to problem solving, this document proposed three additional process standards at each grade level – communication, reasoning, and connections. The “process standards” were further elaborated in NCTM’s updated standards document, *Principles and Standards for School Mathematics*, which was released in 2000. In this document, as in the 1989 publication, problem solving was the first process standard discussed, with the addition of representation as a new process-oriented standard.

The next phase of NCTM’s work to improve mathematics education split into two parallel efforts. First, *Curriculum Focal Points for Prekindergarten through Grade 8*...
Making Reasoning and Sense Making the Focus – p. 2

*Mathematics: A Quest for Coherence* was released in 2007. This document provided grade-by-grade definitions of the content to be taught in each grade in contrast to the grade-band approach in previous documents; however, the process standards were somewhat less prominent. Soon after the release of the *Curriculum Focal Points* document, work began on a companion document for high school entitled *Focus in High School Mathematics: Reasoning and Sense Making*, which was released in 2009. In contrast to the PK-8 document, the high school document maintains a clear focus on the process standards, particularly reasoning and sense making. This document will be the focus of the remainder of this paper.

**Rationale**

The emphasis on reasoning and sense making in *Focus in High School Mathematics* and on process skills more generally throughout past NCTM documents is a direct reflection of clear evidence that the U.S. mathematics curriculum is not preparing students for future success. The Programme for International Student Assessment (2007) shows that students in the United States are not able to apply mathematics to solve problems drawn from real-world contexts. This is part of a long-term pattern of findings showing that U.S. students have difficulty with problems for which there is not a clear, previously-taught solution method (Dossey, 2000). Moreover, students are not receiving the preparation they need for success in a wide range of careers that increasingly require a firm grasp of advanced mathematics and statistics (Ganter & Barker, 2004; American Diploma Project, 2004). Finally, a series of reports argue that the U.S. is no longer meeting the challenge of preparing students for mathematics-intensive fields in science, technology, engineering, and mathematics (cf. Tapping America’s Potential, 2008). A focus on reasoning and sense making will ensure that U.S. students are prepared to meet these challenges.

The document does not define reasoning strictly as logical deduction and proof. Instead, it is defined more broadly as drawing conclusions from evidence, running the gamut from more informal exploration and pre-formal explanations to more formal proofs and explanations. Likewise, sense making is defined as “developing understanding of a situation, context, or concept by connecting it with existing knowledge” (p. 4) and is seen as complementary to reasoning: Reasoning must build on the understanding provided by making sense of a situation, and seeking to justify why something is true will enhance one’s understanding of a situation.
Note that the emphasis on reasoning is not meant to supersede the broader emphasis on process skills of the past 30 years or the more specific emphasis on problem solving as an organizing theme. Indeed, the process standards are all closely intertwined. However, placing reasoning and sense making in the foreground places a primary focus on mathematical thinking, which the document argues should be seen as “the foundation[s] of the NCTM Process Standards. The processes of mathematics… are all manifestations of the act of making sense of mathematics and of reasoning” (NCTM, 2009, p. 5). While the other process standards--problem solving, communication, representation, and connections--might be portrayed as “fuzzy math” that does not promote rigorous mathematics, it is much more difficult to attack mathematical reasoning as unnecessary fluff. Without reasoning there is no mathematics.

Key Messages about Reasoning and Sense Making

The chapters of the document are designed to discuss several key messages, presented at the start of the chapters. These key messages illustrate the breadth of the issues that must be addressed in refocusing the mathematics curriculum on reasoning and sense making.

- A high school mathematics program based on reasoning and sense making will prepare students for citizenship, for the workplace, and for further study. This point has been thoroughly addressed in the previous section.

- Reasoning and sense making should be a part of the mathematics classroom every day. This focus cannot be reserved for only some groups of students, for only some courses, or for only special topics or occasions (“Reasoning and Sense Making Friday!”) where reasoning and sense making are easy to address. A list of “reasoning habits” is identified that should be promoted as an integral part of the study of any lesson. It is also argued that a focus on reasoning and sense making are inherent in achieving mathematical proficiency – including conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, & Findell, 2001).

- Reasoning and sense making are integral to the experiences of all students across all areas of the high school mathematics curriculum. In the U.S., reasoning and proof are often thought to be synonymous with the study of geometry, while other areas of the curriculum, such as formal algebra, are thought to be more procedural and less related
Making Reasoning and Sense Making the Focus

Making Reasoning and Sense Making the Focus – p. 4

to mathematical reasoning. The document exemplifies how reasoning and sense making can be incorporated into the study of all strands of the curriculum – number and measurement, algebraic symbols, functions, geometry, and statistics and probability. “Key elements” for each strand provide curricular emphases that might support that effort, and a series of examples providing an idealized view of reasoning and sense making in the classroom are provided.

- **Mathematical reasoning and sense making must be evident in the mathematical experiences of all students.** Ensuring equitable experiences in mathematics has long been a major focus of NCTM over the past decades (cf. the Equity Principle, NCTM, 2000; NCTM, 2008), and this document emphasizes that reasoning and sense making should not be reserved for the elite. All too often, students who are assigned to “gifted” or “advanced” courses receive a qualitatively different mathematical experience from those relegated to “basic” or “remedial” courses. While advanced students may be thought capable of reasoning and sense making, “lower level” students are often thought to learn best from repeated practice devoid of reasoning. Moreover, student demographics often predict course placements and opportunities for reasoning and sense making—both by race/ethnicity and by socioeconomic status. Finally, teacher expectations play a pivotal role; if teachers believe that some students are incapable of reasoning and sense making, they are less likely to provide the support and encouragement that those students need to succeed in mathematics.

- **Curriculum, instruction, and assessment form a coherent whole to support reasoning and sense making.** All components of the educational system must work in harmony to ensure that students receive the experiences with reasoning and sense making they need for their future success. Teachers must provide instruction that focuses on student reasoning, not just on developing procedures to solve classes of projects. They must provide students with worthwhile tasks that engage students and establish an effective learning environment that values reasoning and sense making. The curriculum must be organized in a manner that includes attention to reasoning and sense making, and that is aligned with expectations from pre-high school mathematics as well as expectations for college mathematics. Assessments all too often emphasize quick recall of facts and procedures at the expense of mathematical reasoning. Efforts
must be made to include reasoning and sense making in large-scale, formal assessments. Moreover, teachers should emphasize formative assessment that attend to reasoning and sense making.

- **Everyone involved must work together to ensure that reasoning and sense making are central foci of high school mathematics programs.** This vision of reasoning and sense making represents a transformation of priorities and methods for high school mathematics. Teachers alone cannot ensure that this will happen. All stakeholders must be involved—including teachers, families, administrators, policymakers, higher education personnel, curriculum designers, and even students. While the document focuses on providing questions for various audiences to explore, a series of short publications addressed to each of these audiences is available on the NCTM website at [www.nctm.org/hsfocus](http://www.nctm.org/hsfocus).

Example – Fuel for Thought

As mentioned above, the document provides a series of examples that exemplify how reasoning and sense making might be incorporated in the high school mathematics classroom. These examples take on a variety of formats (a classroom vignette, sample assessment item, or mathematical exposition) and are meant to provide “an idealized illustration of how reasoning and sense making might unfold” (NCTM, 2009, p. 19). We will explore an example taken from the Number and Measurement strand, focusing on the key element, Reasonableness of Answers and Measurements. This example explores the fuel efficiency of various vehicles. In understanding this example, the reader should be aware that fuel consumption in the U.S. is generally given in “miles per gallon” (MPG)—the miles (1.6 km) one can drive on a gallon (3.8 L) of gasoline. To start the activity, the teacher poses the following quiz to a first-year high school class.

**Quiz time: Which of the following would save more fuel?**

a) Replacing a compact car that gets 34 miles per gallon (MPG) with a hybrid that gets 54 MPG.

b) Replacing a sport utility vehicle (SUV) that gets 18 MPG with a sedan that gets 28 MPG.

c) Both changes save the same amount of fuel. (NCTM, 2009, p. 23)
The document continues with a possible scenario of what might happen in the classroom in response to this task:

The teacher collects the quizzes and asks two of her students to share their answers with the class. The first student responds:

I would say the correct answer is (a). My reasoning is that the change from 34 MPG to 54 is an increase of about 59 percent, but the 18 to 28 MPG change is an increase of only about 56 percent.

The second student responds:

I thought about how much gas it would take to make a 100-mile trip.

Considering the compact car:

\[
\begin{align*}
100 \text{ miles}/54 \text{ MPG} &= 1.85 \text{ gallons used.} \\
100 \text{ miles}/34 \text{ MPG} &= 2.94 \text{ gallons used.}
\end{align*}
\]

So switching from a 34 MPG to a 54 MPG car would save 1.09 gallons of gas.

Looking at the SUV:

\[
\begin{align*}
100 \text{ miles}/28 \text{ MPG} &= 3.57 \text{ gallons used.} \\
100 \text{ miles}/18 \text{ MPG} &= 5.56 \text{ gallons used.}
\end{align*}
\]

So switching from an 18 MPG car to a 28 MPG car saves 1.99 gallons of gas every 100 miles. So you are actually saving more gas by replacing the SUV.

The teacher then asks the class to compare these two responses. After a spirited debate among students who had chosen each of the answers, the class reaches a consensus that both responses had merit, depending on how the problem is interpreted. Although the fuel efficiency increased slightly more for the compact car, the owner would actually save more gallons of gasoline by replacing the SUV if both cars were driven the same number of miles.

The teacher asks the class to explore the relationship of MPG with actual gasoline consumption. After the students work in small groups for a few minutes, the teacher asks one group to show the table of values and graph it has made, as shown below. They explain, “You save less fuel as you go up another 5 MPG over and over. So as we saw in the quiz, MPG can be a little confusing.” (NCTM, 2009, pp. 23-24)
For homework, the teacher asks the students to read the article from the *New York Times* from which the task was developed (Chang, 2008) and to analyze the arguments in a brief essay, concluding with an argument for what unit of measure is best for analyzing fuel efficiency.

By engaging in this problem, students deepen their understanding of number and units of measure through reasoning about and making sense of a problem situation that they may find motivating. In addition, their engagement in the problem will build reasoning habits that they might use in future mathematical analyses: In analyzing a problem, it reinforces the habit of seeking patterns and relationships. It also builds important habits in reflecting on a solution, including interpreting a solution and how it answers the problem; reconciling different approaches to the problem, including those proposed by others; and refining arguments so that they can be effectively communicated.

**Tips for Teachers**

While idealized, the description of the classroom highlights a number of important points about how teachers can encourage students to personally engage in reasoning and sense making. Merely observing the teacher’s reasoning and trying to model it in the future is not likely to build students’ mathematical reasoning ability. However, the teacher has an important role to play in selecting engaging tasks and asking probing questions that encourage the students to persevere. The following is a preliminary list of “tips” that is presented in the document to help teachers think more carefully about their critical role; a number of these were highlighted in the preceding example.
• Provide tasks that require students to figure things out for themselves.
• Ask students to restate the problem in their own words, including any assumptions they have made.
• Give students time to analyze a problem intuitively, explore the problem further by using models, and then proceed to a more formal approach.
• Resist the urge to tell students how to solve a problem when they become frustrated; find other ways to support students as they think and work.
• Ask students questions that will prompt their thinking—for example, “Why does this work?” or “How do you know?”
• Provide adequate wait time after a question for students to formulate their own reasoning.
• Encourage students to ask probing questions of themselves and one another.
• Expect students to communicate their reasoning to their classmates and the teacher, orally and in writing, through using proper mathematical vocabulary.
• Highlight exemplary explanations, and have students reflect on what makes them effective.
• Establish a classroom climate in which students feel comfortable sharing their mathematical arguments and critiquing the arguments of others in a productive manner.

In the “Fuel for Thought” example, the teacher presents an engaging task and asks the students to “figure things out for themselves.” Throughout, she gives students ample time to think about the problem and to come up with their own solutions. She asks questions that extend their thinking. Paramount to the success of this exploration is a classroom climate in which students are expected to engage in reasoning and sense making. Students are expected to both share their own thinking and explore the thinking of their classmates in a nonjudgmental manner that values their reasoning and sense making. Classrooms like this are likely to help students not only build mathematical content knowledge but also to deepen their ability to reason and make sense of mathematics.

Next Steps

Refocusing U.S. high school mathematics classes on reasoning and sense making is a major endeavor that will require years of concerted effort by all parties involved. NCTM is developing a comprehensive plan to support this effort. In addition to the primary Reasoning and
Sense Making document, a series of “topic books” is being released that provides additional guidance in particular content areas (algebra, geometry, and probability and statistics) and in addressing particular issues, including the use of technology and ensuring equitable access of all students to reasoning and sense making. Links to these documents, along with other supporting brochures and resources, can be found at www.nctm.org/hsfocus. A number of initiatives have been created to support the effort, including the development of a library of video examples of classrooms engaged in reasoning and sense making and the design of a major conference to be held the summer of 2011. A bank of tasks that support reasoning and sense making is also in development. While the initial focus of these endeavors is on high school mathematics, the group planning the larger project recognizes that the goals for high school mathematics cannot be realized without a corresponding refocus in grades PK-8, and efforts will be made to extend the initiative beyond the high school years in the future. Reasoning and sense making should be a focus of mathematics instruction across all the grades, not just in the high school years.

A recent development in the U.S. has the potential to extend the impact of this effort. In July 2010 the Common Core State Standards Initiative (CCSSI), comprised of national educational organizations, released standards for mathematics for grades K-12 that are being considered for adoption by states across the union. While these standards have been developed at a national level, they are not mandatory. To date, 30 of the 50 states have voluntarily adopted these standards which currently encompass both mathematics and language arts. In addition to spelling out detailed content standards, the Common Core State Standards for Mathematics (CCSSI, 2010) specify a Mathematical Practices Standard which is in general alignment with the recommendations of the Focus in High School Mathematics document. This standard establishes the importance of mathematical reasoning as an integral and necessary part of the mathematics curriculum. Four major mathematics education organizations have united to support the implementation of the common core state standards (NCTM et al., 2010), and the political clout represented in the common core state standards may help to provide increased emphasis on reasoning and sense making and the process standards as they are implemented. Thus, the common core state standards provide a window of opportunity to build on NCTM’s efforts over the past decades to improve the quality and depth of mathematics instruction in the U.S. through an emphasis on process skills at the high school level and indeed across all grade levels.
References


Common Cores State Standards Initiative. *Common Core State Standards for Mathematics,*


http://www.nctm.org/standards/content.aspx?id=26088